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Electrohydrodynamic stability of a liquid jet

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Abstract. The stability of a liquid jet under a time-dependent electric field has been investigated. It is found that, for perturbations having large wavelengths, the electric field has a stabilizing effect. For perturbations having small wavelengths, there are resonance modes causing excitation of the perturbed surface of the jet.

1. Introduction

The problem of stability of a liquid jet under the influence of an electric field has been investigated by many authors in recent years. Glonti (1958) and Nayyar and Murty (1960) have studied the effect of an axial electrostatic field on the stability of a dielectric liquid jet and found that the electric field has a stabilizing effect. Reynolds (1965), Crowley (1965) and Melcher (1963) have investigated the stability of a jet under a radial time-varying electric field, and found that under certain conditions the field may have a stabilizing or destabilizing effect. We shall discuss here the effect of a time-varying axial field on the stability of a jet. It is well known that in the absence of an electric field the critical stable length λ_c of a cylindrical liquid jet is $2\pi R$, where R is the radius of the jet. Nayyar and Murty (1960) have shown that the critical length of the jet increases in the presence of an axial electric field. For a given value of the applied electric field the jet is stable for all modes of perturbation with wave numbers x (in units of the radius of the jet) exceeding a certain critical value x_c. In the present case the time-dependent axial field will cause some modification to this criteria, namely for small values of x less than one, the stability argument essentially remains unaffected while, for large values of x greater than one, the jet will not always be stable as before but there will be bounded regions of stability.

2. Outline of the procedure

We shall consider an infinite dielectric liquid cylinder of radius R with density and dielectric constant ρ_i and ϵ_i respectively. The cylinder is surrounded by another dielectric medium having density and dielectric constant ρ_0 and ϵ_0 respectively (henceforth the subscripts or superscripts i, o will refer to inner or outer fluid respectively). The cylinder is subject to a time-varying axial electric field of the type

$$\bar{E} = E^*\mathrm{Re}\{\exp(\mathrm{i}\omega t)\}\mathbf{1}_z \qquad (2.1)$$

where $\mathbf{1}_z$ is the unit vector in the z direction, which is taken as the axis of the cylinder (using cylindrical polar coordinates).

If we allow a small departure from the equilibrium state, the linearized equation of motion takes the form

$$\rho\frac{\partial \boldsymbol{v}}{\partial t} = -\nabla\Pi \qquad (2.2)$$

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where

$$\Pi = p - \rho \left(\frac{\partial \epsilon}{\partial \rho}\right)_T \bar{\boldsymbol{E}} \, . \, \delta \boldsymbol{E} \tag{2.3}$$

and p, $\delta \boldsymbol{E}$ and $\boldsymbol{v}$ are, respectively, the perturbations in pressure, electric field and the velocity of the jet. The second term in the right-hand side of equation (2.3) is due to electrostriction. We shall assume that the jet and the surrounding medium are incompressible fluids, so that the equation of continuity is

$$\nabla \, . \, \boldsymbol{v} = 0. \tag{2.4}$$

From equations (2.2) and (2.4) we get

$$\nabla^2 \Pi = 0. \tag{2.5}$$

We shall make use of the quasi-static approximation, i.e. the effect of displacement currents will be neglected. The perturbed potential will therefore satisfy Laplace's equation

$$\nabla^2 \delta \phi = 0 \tag{2.6}$$

where

$$\delta \boldsymbol{E} = -\nabla \delta \phi. \tag{2.7}$$

The surface deformation is assumed to be of the form

$$\Sigma = R + \gamma(t) \exp\{\mathrm{i}(kz + m\varphi)\} \tag{2.8}$$

where $\gamma(t)$ is a small function of time to be solved.

The solutions for equations (2.5) and (2.6) can be put in the following forms

$$\Pi^{\mathrm{i}} = B_1(t) \frac{\mathrm{I}_m(kr)}{\mathrm{I}_m'(x)} \exp\{\mathrm{i}(kz + m\varphi)\} \tag{2.9}$$

$$\Pi^{\mathrm{o}} = B_2(t) \frac{\mathrm{K}_m(kr)}{\mathrm{K}_m'(x)} \exp\{\mathrm{i}(kz + m\varphi)\} \tag{2.10}$$

where

$$x = kR$$

$$\delta\phi^{\mathrm{i}} = A_1(t)\gamma(t)\mathrm{K}_m(x)\mathrm{I}_m(kr) \exp\{\mathrm{i}(kz + m\varphi)\} \tag{2.11}$$

$$\delta\phi^{\mathrm{o}} = A_2(t)\gamma(t)\mathrm{I}_m(x)\mathrm{K}_m(kr) \exp\{\mathrm{i}(kz + m\varphi)\} \tag{2.12}$$

where I_m and K_m are the modified Bessel functions of pure imaginary arguments of order m of the first and second kind respectively. B_1, B_2, A_1 and A_2 are time-dependent constants of integration which are to be evaluated by making use of the following appropriate boundary conditions:

(i) The normal component of the velocity at the interface should be compatible with the assumed surface deformation defined by equation (2.8).

(ii) The potential $\delta\phi$ should be continuous at the interface.

(iii) The normal component of the electric displacement should be continuous at the interface, namely,

$$\epsilon_1 \boldsymbol{N} \, . \, \boldsymbol{E}^{\mathrm{i}} = \epsilon_0 \boldsymbol{N} \, . \, \boldsymbol{E}^{\mathrm{o}} \tag{2.13}$$

where $\boldsymbol{E}^{\mathrm{i},\mathrm{o}}$ is the total electric field and $\boldsymbol{N}$ is the unit normal vector to the interface, pointing outwards.

(iv) The normal component of the momentum tensor should be continuous at the interface, namely,

$$\Pi^{i}-\Pi^{o} = \tfrac{1}{2}\epsilon_{i}\{(E^{i})^{2}-\bar{E}^{2}\} -\tfrac{1}{2}\epsilon_{o}\{(E^{o})^{2}-\bar{E}^{2}\} + \frac{\gamma T}{R^{2}}(x^{2}+m^{2}-1)\exp\{i(kz+m\varphi)\} \tag{2.14}$$

where T is the surface tension.

Applying condition (i) to (2.9) and (2.10) we get

$$B_1 = -\frac{\rho_i\gamma''}{k}, \qquad B_2 = -\frac{\rho_o\gamma''}{k} \tag{2.15}$$

where

$$\gamma'' = \frac{d^2\gamma}{dt^2}.$$

Conditions (ii) and (iii) when applied to equations (2.10) and (2.11) yield

$$A_1(t) = A_2(t) = -\frac{iE^{*}\exp(i\omega t)\,(\epsilon_i-\epsilon_o)}{\epsilon_i K_m(x) I_m'(x)-\epsilon_o K_m'(x) I_m(x)}. \tag{2.16}$$

Substituting from equations (2.15), (2.16) and (2.9)–(2.12) into equation (2.14) and simplifying we obtain

$$\frac{d^2\gamma}{dt^2}\left(\frac{\rho_i I_m K_m'-\rho_o I_m' K_m}{I_m' K_m'}\right) + \left\{\frac{Tx}{R^3}(x^2+m^2-1)+\frac{x^2E^{*2}}{R^2}\frac{(\epsilon_i-\epsilon_o)^2 K_m I_m\cos^2\omega t}{\epsilon_i\,.\,K_m I_m'-\epsilon_o K_m' I_m}\right\}\gamma = 0. \tag{2.17}$$

3. Analysis of stability

The solution of equation (2.17) will decide the criterion of stability of the jet. Accordingly, the jet will be stable if the solution for $\gamma(t)$ will remain bounded as $t \to \infty$, otherwise it is unstable. Let us use the following notations for simplicity

$$b = \frac{Tx}{\omega^2 R^3}\frac{(x^2+m^2-1)I_m' K_m'}{\rho_i I_m K_m'-\rho_o I_m' K_m} \tag{3.1}$$

$$h^2 = -\frac{x^2E^{*2}}{\omega^2R^2}\frac{(\epsilon_i-\epsilon_o)^2 K_m K_m' I_m I_m'}{(\epsilon_i K_m I_m'-\epsilon_o K_m' I_m)(\rho_i K_m' I_m-\rho_o K_m' I_m')}. \tag{3.2}$$

From (3.1), (3.2) and (2.17) we obtain

$$\frac{d^2\gamma}{d\eta^2}+(b-h^2\cos^2\eta)\gamma = 0 \tag{3.3}$$

where

$$\eta = \omega t.$$

Equation (3.3) is the Mathieu differential equation, which can be put in the equivalent form

$$\frac{d^2\gamma}{d\eta^2}+(a-2q\cos 2\eta)\gamma = 0 \tag{3.4}$$

where

$$a = \left\{\frac{Tx}{\omega^2 R^3}(x^2+m^2-1) + \frac{xE^{*2}}{2\omega^2 R^2}\frac{\mathrm{I}_m\mathrm{K}_m(\epsilon_i-\epsilon_o)^2}{\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m}\right\}\frac{\mathrm{K}_m'\mathrm{I}_m'}{\rho_i\mathrm{K}_m'\mathrm{I}_m-\rho_o\mathrm{K}_m\mathrm{I}_m'} \quad (3.5)$$

and

$$q = -\frac{x^2E^{*2}}{4\omega^2R^2}\frac{\mathrm{K}_m'\mathrm{K}_m\mathrm{I}_m\mathrm{I}_m'(\epsilon_i-\epsilon_o)^2}{(\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m)(\rho_i\mathrm{K}_m'\mathrm{I}_m-\rho_o\mathrm{K}_m\mathrm{I}_m')}. \quad (3.6)$$

The condition for stability reduces to the problem of the bounded regions of the Mathieu functions which (McLachlan 1947) gives the condition of stability as

$$|\Delta(0)\sin^2\tfrac{1}{2}\pi a^{1/2}| \leqslant 1 \quad (3.7)$$

where $\Delta(0)$ is the Hill's determinant.

As a limiting case when $\omega \to 0$ we obtain the solution for the static case and equation (2.17) reduces to

$$\frac{\rho_i\mathrm{K}_m'\mathrm{I}_m-\rho_o\mathrm{K}_m\mathrm{I}_m'}{\mathrm{I}_m'\mathrm{K}_m'}\frac{\mathrm{d}^2\gamma}{\mathrm{d}t^2}\left\{\frac{Tx}{R^3}(x^2+m^2-1)+\frac{x^2E^{*2}}{R^2}\frac{(\epsilon_i-\epsilon_o)^2\mathrm{I}_m\mathrm{K}_m}{\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m}\right\}\gamma = 0 \quad (3.8)$$

which is a linear differential equation with constant coefficients and can be satisfied by $\gamma = \exp(\sigma t)$ where

$$\sigma^2 = -\frac{\mathrm{I}_m'\mathrm{K}_m'}{\rho_i\mathrm{K}_m'\mathrm{I}_m-\rho_o\mathrm{K}_m\mathrm{I}_m'}\left\{\frac{Tx}{R^3}(x^2+m^2-1)+\frac{x^2E^{*2}}{R^2}\frac{(\epsilon_i-\epsilon_o)^2\mathrm{K}_m\mathrm{I}_m}{\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m}\right\}. \quad (3.9)$$

So stability occurs if

$$\frac{Tx}{R}(x^2+m^2-1)+\frac{x^2E^{*2}(\epsilon_i-\epsilon_o)^2\mathrm{K}_m\mathrm{I}_m}{\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m} \geqslant 0. \quad (3.10)$$

For axisymmetric perturbation ($m = 0$), inequality (3.10) becomes

$$\frac{T}{R}(x^2-1)+\frac{xE^{*2}(\epsilon_i-\epsilon_o)^2\mathrm{K}_0\mathrm{I}_0}{\epsilon_i\mathrm{K}_0\mathrm{I}_1+\epsilon_o\mathrm{K}_1\mathrm{I}_0} \geqslant 0 \quad (3.11)$$

which is the same as that obtained earlier by Nayyar and Murty (1960).

In general the analysis of inequality (3.7) is rather complicated due to the infinite Hill's determinant. An approximate formula for stability given by Morse and Feshbach (1953) may be used for small values of $|h^2|$ or q which is a good approximation to high-frequency fields. According to this assumption the jet will be stable if

$$h^4 - 16(1-b)h^2 + 32b(1-b) \geqslant 0. \quad (3.12)$$

The above inequality is a quadratic expression and can be investigated using elementary theory of equations. If we write

$$\left.\begin{aligned} \Delta &= +\{32(1-b)(2-3b)\}^{1/2} \\ \epsilon^* &= \frac{(\epsilon_i-\epsilon_o)^2\mathrm{I}_m\mathrm{K}_m}{\epsilon_i\mathrm{K}_m\mathrm{I}_m'-\epsilon_o\mathrm{K}_m'\mathrm{I}_m} \\ \text{and} \quad \rho^* &= \frac{\mathrm{I}_m'\mathrm{K}_m'}{\rho_i\mathrm{K}_m'\mathrm{I}_m-\rho_o\mathrm{K}_m\mathrm{I}_m'} \end{aligned}\right\} \quad (3.13)$$

the analysis can be classified in two categories:

(i) For those modes for which $b > 0$ (which means that $x^2+m^2 \geqslant 1$) the analysis can be summarized as follows:

(I) For $0 < b < 1$, the jet will be stable for all modes.

(II) For $b > 1$, the jet will be stable if

$$\frac{E^{*2}}{\omega^2} \geqslant \frac{R^2}{x^2\rho^*\epsilon^*}\{8(b-1)+\Delta\}. \tag{3.14}$$

It is clear that this result is different from the case for electrostatic fields applied to the jet, since the latter case admits stability for all modes for which $x^2+m^2 \geqslant 1$ (i.e. $b \geqslant 0$). If we define the crical amplitude E_c^* of the applied electric field as

$$E_c^{*2} = \frac{\omega^2 R^2}{x^2\rho^*\epsilon^*}\{8(b-1)+\Delta\} \tag{3.14a}$$

then from inequality (3.14) we find that for $x \gg 1$, the system is always stable for $E^{*2} \geqslant E_c^{*2}$ and unstable for $E^{*2} < E_c^{*2}$. For the given parameters of the jet listed in table 1 and for the frequency of the applied field, $\omega = 120\pi$ Hz, we find that the jet is always stable for $x < x_c(\simeq 27)$. The dependence of E_c^* on x ($x > x_c$) is given in table 1 for axisymmetric modes ($m = 0$).

Table 1

x	E_c^* (V cm^{-1})
27	5205·90
28	7493·70
29	9004·60
30	10192·09
31	11186·76
32	12049·34
33	12814·37
34	13503·88
35	14132·98
36	14712·57
37	15250·84

Parameters of jet: $\epsilon_i = 81$, $\epsilon_o = 1$, $R = 2$ cm, $T = 74$ dyn cm^{-1}, $\rho_i = 1$ g cm^{-3}, $\rho_o = 0$, $m = 0$.

For the typical values 800 Hz and 900 Hz of ω we find that the values of x_c are approximately 42 and 47 respectively.

In our case, if E^* does not satisfy inequality (3.14) for $b > 1$, the jet will be unstable and the time-varying field has a destabilizing effect.

(ii) If $b < 0$, according to inequality (3.12), stability can occur only when

$$\frac{E^{*2}}{\omega^2} \geqslant \frac{R}{x^2\epsilon^*\rho^*}\{8(b-1)+\Delta\}. \tag{3.15}$$

Except for the frequency dependence, this condition is similar to that in the electrostatic case. We can similarly define the critical amplitude E_c^* (equation (3.14a)) such that for a given value of x ($x < 1$), the jet is stable for $E^* \geqslant E_c^*$, and unstable for $E^* < E_c^*$. In table 2 we have given the values of E_c^* for various values of ω and x.

Table 2

	E_c^{static} (V cm^{-1})			E_c^* (V cm^{-1})		
$x \setminus \omega$(Hz)	0	200	120π	600	800	950
0·1	724·32	1267·33	1520·88	2032·25	2841·93	3374·79
0·2	581·78	870·90	932·14	1174·32	1212·83	1314·75
0·3	531·47	766·92	795·92	856·13	957·57	1166·66
0·4	494·99	711·70	711·74	788·86	833·46	808·11
0·5	458·50	653·34	667·15	728·57	758·82	780·37
0·6	417·09	592·64	612·59	641·28	692·90	656·38
0·7	367·39	524·71	539·63	546·44	583·68	625·33
0·8	304·83	433·59	454·86	443·02	458·86	552·63
0·9	218·76	312·16	320·85	341·95	375·16	385·78
1·0	0	0	0	0	0	0

The jet dimensions are the same as given in table 1.

From table 2 we see that for a given $x \leqslant 1$ the values of E_c^* increase with increasing ω, and, further, for any value of $x \leqslant 1$, $E_c^*(\omega) > E_c^{\text{static}}$, although the dependence of E_c^* on x is similar to that of the corresponding electrostatic case. One can justify this similarity if one studies the characteristic curve of the Mathieu functions in the

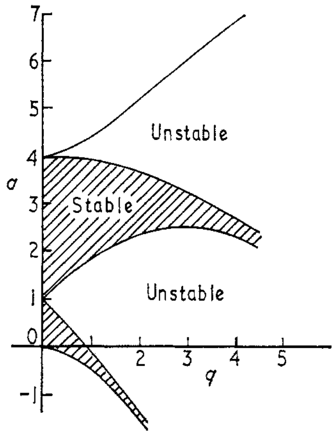

Figure 1. The characteristic curves of the Mathieu function for the first region of instability in the (a,q) plane. Shaded regions are regions of stability.

(a,q) plane (figure 1). Concentrating on the first region of stability $(0 < a < 1)$, and for small values of q, we obtain for $a > 0$ the inequality

$$E^{*2} + \frac{2T}{xR\epsilon^*}(x^2+m^2-1) > 0. \tag{3.16}$$

This may be considered as an approximation to inequality (3.15), and it is similar to the electrostatic condition. The field in this case has a stabilizing effect.

If we let

$$f(h^2, b) \equiv h^4 - 16h^2(1-b) + 32b(1-b) \tag{3.17}$$

the stability condition given by inequality (3.12) will be

$$f(h^2, b) \geqslant 0. \tag{3.18}$$

The curve $f(h^2, b) = 0$ in the (b, h^2) plane represents a hyperbola which touches the lines $b = \frac{2}{3}$ and $b = 1$ and intersects the b axis at the origin and $b = 1$ (figure 2). Since

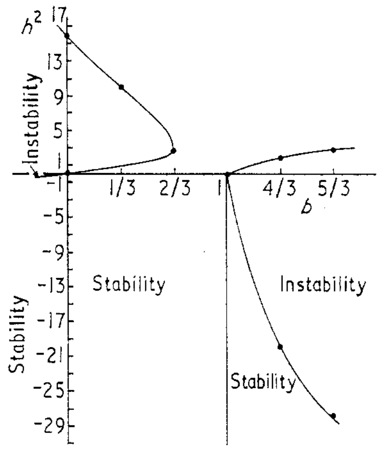

Figure 2. Approximate regions of stability defined by the hyperbola represented by the equations $f(h^2, b) = 0$. The regions appear in the lower half-plane.

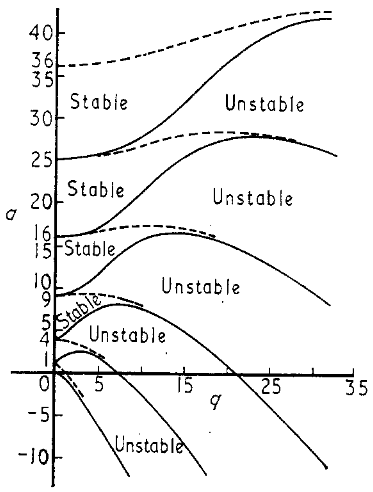

Figure 3. The different regions of stability and instability of the characteristic curves of the Mathieu functions.

h^2 is always negative in equation (3.2), we shall be interested in the lower half-plane only (the regions of stability and instability are shown in figure 2). This figure may be considered as the approximated first region of instability of the graph of the characteristic curves of the Mathieu functions (figure 3).

4. Resonance modes

If we examine generally the characteristic curves of the Mathieu functions (figure 3) we see that the regions of instability get narrower as we move towards the a axis until they tend to points on the a axis at values of a equal to n^2 where $n = 1, 2, 3 \ldots$.

In these narrow regions, the electric field gives energy to the disturbed fluid, the amplitude of the disturbance grows exponentially and a state of resonance is reached. As the value of q decreases, the resonance becomes sharper and the points of sharp resonance are given by

$$\rho^* \left\{\frac{Tx}{\omega^2 R^3}(x^2+m^2-1)+\frac{x^2 E^{*2}\epsilon^*}{2\omega^2 R^2}\right\} = n^2, \qquad n = 1, 2, 3, \ldots . \tag{4.1}$$

For a given n, the roots of equation (4.1) give the resonance wave numbers in terms of E^*, ω and the various parameters of the jet.

The resonance occurs when the frequency of the applied field satisfies the relation

$$\sigma' = n\omega, \qquad n = 1, 2, 3, \ldots . \tag{4.2}$$

where

$$\sigma'^2 = \frac{\rho^*}{2R^2}\left\{\frac{2Tx}{R}(x^2+m^2-1)+x^2E^{*2}\epsilon^*\right\}. \tag{4.3}$$

Equation (4.3) is similar to the dispersion relation for a jet in the presence of a uniform electrostatic field of strength E^* and subject to a perturbation of the form $\exp\{i(\sigma' t + kz + m\varphi)\}$.

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